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Admittance/Fourier series revisited: understanding periodic heat flows

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Abstract

Thermal modelling of buildings typically involves the use of software programs that are highly accurate but complex. As such many users do not have a good "feel" for how heat flows in and out of a building. The simplest type of manual calculation method is a steady state model which allows some insight into the flow of heat in a building. However modelling of thermal storage in building elements with mass is seen to be too difficult to be solved readily and as such complex thermal software programs are utilised. In the late 1960s and early 1970s the admittance method was developed which calculated quite accurately the thermal response of building elements with mass. Typically the thermal response of a building to 24 hour cyclic inputs - temperatures and solar radiation – was calculated. This sort of calculation was also extended to higher frequencies utilising a more accurate Fourier series representation of temperatures and solar radiation. However this approach was soon overtaken by more complex computer based models which delivered greater modelling complexity and accuracy but tended to obscure the underlying physical processes. This paper re-examines the admittance/Fourier method as a pathway to enhancing understanding of the response of buildings to fluctuating temperatures and solar radiation. A simplified representation of yearly ambient temperature in terms of only three terms: a constant, a yearly and daily frequency – allows a very simple model of building to be developed. This approach can allow building designers rapid insight into the performance of various materials and designs and in addition enhance their understanding of the fundamental physical processes involved.

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Keywords: Admittance; Fourier series; superposition; periodic; heat transfer

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1. Introduction

The analysis of heat flowing in and out of a building can be carried out in a number of ways. Starting with simple steady state analysis and utilizing the thermal resistance of building elements (or their inverse, *U*-values) – steady state calculations allow some insight into the thermal performance of a building design. However once time varying external temperatures, solar radiation and building elements with thermal capacitance need to be analyzed most researchers and practitioners utilize a sophisticated, modern computer based thermal analysis program to tackle the problem. Of course this latter approach has been shown to be accurate in calculating a wide range of heat transfer processes that are complex and necessary in designing buildings. However it seems that there is little understanding of the middle ground between these two cases of a simple steady state calculation and a complex, thermal computer based program. As such inexperienced designers, practitioners and researchers are often at the mercy of accepting whatever results a complex thermal program produces. On the other hand those with many years' experience may have a better physical "feel" for these complex calculations and be able to avoid situations where there are errors in the results they are obtaining perhaps due to errors in their input parameters or if there is an error in the program itself.

This paper seeks to revisit techniques for the analysis of periodic and time varying heat flows that were in vogue before complex computer calculations took over the field. The admittance approach was strongly advanced by researchers predominantly in the UK, such as Milbank and Harrington [1], Davies [2], as well as Athienitis and Santamouris [3], and Muncey [4]. It is not the intent of this paper to develop yet another approach to try and compete with existing complex thermal programs. Instead this paper seeks to elucidate and explore the admittance method of analyzing periodic heat flow in order to yield greater insight or "feel" for how time-varying heat flows actually occur through building elements for simplified cases.

1.1. Periodic solution of the heat equation

To analyse periodic heat flow in a building involves analysing the response of the building envelope to sinusoidal temperatures or irradiances. Due to Fourier's theory – any time varying function can be represented by a summation of different frequency cosine and sine waves [3]. Hence, any time varying temperatures, solar irradiance, mechanical heating or cooling, or any other heat flow can be analysed. . It is useful to define for any temperature, T as being equal to the sum of the steady state temperature, $\bar{T}$ and the periodic component $\tilde{T}$. That is, $T = \bar{T} + \bar{\tilde{T}}$. Similarly for heat flows, $\dot{Q} = \bar{Q} + \tilde{Q}$.

Consider now a uniform wall of thickness L, density ρ, thermal conductivity k and specific heat capacity c_p. The periodic solution of the heat equation [5] as a function of distance x and time t, for the periodic temperature inside the wall $\tilde{T}(x,t)$ is the real part of:

$$\tilde{T}(x,t) = \tilde{T}_o(t)\frac{cosh(\gamma(L-x)) + \frac{h_i}{k\gamma}sinh(\gamma(L-x))}{\left(1+\frac{h_i}{h_o}\right)cosh(\gamma L) + \left(\frac{h_i}{k\gamma}+\frac{k\gamma}{h_o}\right)sinh(\gamma L)}, \quad (1)$$

where $\tilde{T}_o(t)$ is the sinusoidal, outside temperature ($|T_o|e^{j\omega t}$), $|T_o|$ is the amplitude of the outside temperature, and the inside temperature $\tilde{T}_i$ is set to zero. The terms h_i and h_o are the convection heat transfer coefficients at the inside and outside surface of the wall respectively, and γ is given by:

$$\gamma = \sqrt{\frac{\omega\rho c_p}{2k}}(1+j), \quad (2)$$

where ω is the angular frequency of the outside temperature (and is related to the period, P of the oscillation by the equation $P = 2\pi/\omega$), $j = \sqrt{-1}$, and hence γ is a complex quantity. Another useful quantity to define is the effective thickness, δ [5]:

$$\delta = \sqrt{2k/\omega\rho c_p}. \quad (3)$$

Shown in Fig. 1 is the variation of the periodic temperature inside a 600 mm thick concrete wall for various times and for periods of (a) 24 h and (b) 8760 h in response to a 4 degree amplitude, sinusoidal, outside temperature variation. The parameters are: L = 600 mm, ρ = 1900 kg/m^3, thermal conductivity k = 1.1 W/m.K, specific heat capacity c_p, = 1000 J/kgK, h_i = 7.7 W/m^{2}K and h_o = 25 W/m^{2}K (as per [8]). The outside surface of the wall occurs at x = 0. Note the damped, sinusoidal variation of the temperature profile for the 24 hour case (Fig. 1a). This occurs due to the charging and discharging of the thermal capacitance of the wall in response to the periodic temperature oscillation. For typical thicknesses of a concrete wall (say 100 – 200 mm thick) the temperature oscillation is still quite significant in comparison to the external temperature oscillation. For infinitely thick walls the temperature variation stays within an envelope that decays with distance proportional to: exp($-x/\delta$). For the wall shown in Fig. 1(a), the effective thickness δ = 0.13 m. Note that after a distance of 3δ, the temperature oscillations inside the wall are significantly damped. However for a yearly period the temperature profile in the wall is essentially linear (Fig. 1(b)). That is, the wall responds to a yearly periodic temperature source in the same way as it would for a steady state temperature difference across a wall. This is because for a period of a year (8760 h) the effective distance δ is 2.4 m.

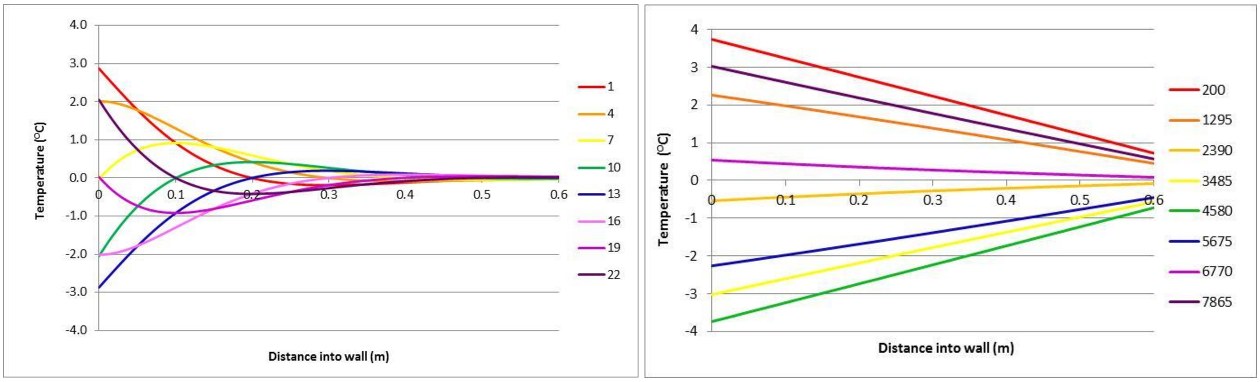


Fig. 1. Temperature inside a wall as a function of distance and time for a) a 24 hour period and b) a yearly period. Different curves correspond to a specific time as indicated by the legend a) 1,4,6..21 hours and b) 200, 1295,… 7865 hours.

1.2. Transfer and self-admittance

From the above periodic solution, the heat transfer characteristics of a building element of area A can be characterized using a two port analysis – borrowed from the analysis of electrical circuits (see for example Carslaw and Jaeger [6], and Athienitis and Santamouris [3]).

$$\begin{bmatrix} \tilde{T}_{wo} \\ \tilde{Q}_o \end{bmatrix} = \begin{bmatrix} cosh(\gamma L) & \frac{sinh(\gamma L)}{Ak\gamma} \\ Ak\gamma\, sinh(\gamma L) & cosh(\gamma L) \end{bmatrix} \begin{bmatrix} \tilde{T}_{wi} \\ \tilde{Q}_i \end{bmatrix} \quad (4)$$

where $\tilde{T}_{wo}$ and $\tilde{T}_{wi}$ are the periodic temperatures at the outside and inside surface of the wall and $\tilde{Q}_o$ and $\tilde{Q}_i$ are the rates of heat flow at the outside and inside wall surface respectively.

The matrix form of Eqn. 3 means that walls or elements with multiple layers can be easily described using a matrix for each layer. By multiplying the matrices together, this yields a single 2 x 2 matrix to describe the complete response of all the layers of materials [6,7,8]. Thermal resistances at external surfaces R_s are also represented by a suitable matrix. Hence for a wall with external, R_{so} and internal R_{si} thermal resistances associated with convection heat transfer, the appropriate matrix equation can be written as:

$$\begin{bmatrix}\tilde{T}_o\\ \tilde{\dot{q}}_o\end{bmatrix}=\begin{bmatrix}1 & R_{so}\\ 0 & 1\end{bmatrix}\begin{bmatrix}cosh(\gamma L) & sinh(\gamma L)/k\gamma\\ k\gamma sinh(\gamma L) & cosh(\gamma L)\end{bmatrix}\begin{bmatrix}1 & R_{si}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{T}_i\\ \tilde{\dot{q}}_i\end{bmatrix}=\begin{bmatrix}M_1 & M_2\\ M_3 & M_4\end{bmatrix}\begin{bmatrix}\tilde{T}_i\\ \tilde{\dot{q}}_i\end{bmatrix} \tag{5}$$

where $\tilde{T}_o$and $\tilde{T}_i$ are the outside and inside temperatures respectively and $\tilde{\dot{q}}_o$ and $\tilde{\dot{q}}_i$ are the rate of heat flow per unit area at the appropriate surface of the wall and ***M*** is a 2 x 2 matrix that describes the complete response of the wall system including surface resistances (in agreement with [3,6,7,8] with slight sign differences only due to different definitions of the direction of positive heat flow). The direction of positive heat flow, for this two port description of the wall, is shown in Fig. 2.

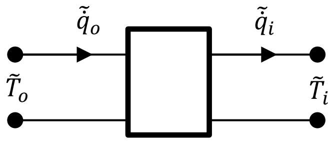


Fig. 2. A two port representation of the periodic heat transfer associated with a wall.

To fully analyze periodic heat flows $\tilde{\dot{q}}$ it is useful to consider two cases. The first is to consider heat flows due to $\tilde{T}_o$ with $\tilde{T}_i$ = 0. For this case it is useful to define $\tilde{\dot{q}}_{in}=\tilde{\dot{q}}_i$. Then the transfer admittance Y_t is defined by the equation [3, 8]:

$$\tilde{\dot{q}}_{in}=Y_t\tilde{T}_o \tag{6}$$

That is, Y_t is the admittance that determines $\tilde{\dot{q}}_{\mathrm{in}}$, the periodic heat flow per unit area at the internal surface of a wall due to an *external* periodic temperature variation $\tilde{T}_o$ with $\tilde{T}_i$ = 0. Note that positive values of $\tilde{\dot{q}}_{\mathrm{in}}$ correspond to heat flowing from the internal wall surface *into* the internal temperature node.

The second case considers heat flows due to $\tilde{T}_i$ with $\tilde{T}_o$ = 0. For this case it is useful to define, $\tilde{\dot{q}}_{out}=-\tilde{\dot{q}}_i$. Then the self-admittance Y_s is defined by [3, 8]:

$$\tilde{\dot{q}}_{out}=Y_s\tilde{T}_i \tag{7}$$

That is Y_s is the admittance that determines $\tilde{\dot{q}}_{\mathrm{out}}$, the periodic heat flow per unit area at the internal surface of a wall due to an *internal* periodic temperature variation $\tilde{T}_i$ with $\tilde{T}_o$ = 0. Note that positive values of $\tilde{\dot{q}}_{out}$ correspond to heat flowing *out* from the internal temperature node and into the internal wall surface.

The literature is not often explicit in explaining exactly why Y_s and Y_t are defined in terms of the heat flow due to one temperature source whilst the other is zeroed. However it is a standard approach for analyzing electrical circuits utilizing the superposition theorem [9] and this issue will be discussed more fully below. Note that Y_t and Y_s are complex quantities (with magnitude and phase) and can be derived from Eqn. 5 in terms of the elements of the matrix ***M*** which gives in the general case: $Y_t=1/M_2$ and $Y_s=M_1/M_2$. Explicitly, from Eqn. 5, for a wall of a single material with surface thermal resistances then:

$$Y_t=[(R_{so}+R_{si})cosh(\gamma L)+sinh(\gamma L)/k\gamma+R_{so}R_{si}cosh(\gamma L)]^{-1} \tag{8}$$

$$Y_s=\frac{cosh(\gamma L)+R_{so}sinh(\gamma L)/k\gamma}{(R_{so}+R_{si})cosh(\gamma L)+sinh(\gamma L)/k\gamma+R_{so}R_{si}cosh(\gamma L)} \tag{9}$$

Note that for the case where $R_{\mathrm{so}}=R_{\mathrm{si}}=0$ then $Y_s=k\gamma\coth(\gamma L)$. For the case where $R_{\mathrm{so}}\to\infty$ and $R_{\mathrm{si}}=0$ then $Y_s=k\gamma\tanh(\gamma L)$. As γ is a function of the periodic angular frequency via Eqn. 2, then the admittances need to be computed for every frequency considered for any calculation.

1.3. Superposition

Superposition allows the calculation of heat flows for any thermal (or electrical) circuit. For a circuit such as Fig. 2 – with heat flows due to two time varying periodic temperatures $\tilde{T}_o$ and $\tilde{T}_i$, the superposition principle states that the net heat flow $\tilde{\dot{q}}_i$ is the sum of $\tilde{\dot{q}}_{out}$ and $\tilde{\dot{q}}_{in}$. That is the sum of the heat flows which occur when considering the flow of heat $\tilde{\dot{q}}_i$ due to only one temperature source at a time while the other is set to zero [9]. Hence, as $\tilde{\dot{q}}_{in}$ and $\tilde{\dot{q}}_{out}$ have opposite signs, the net heat flow is:

$$\tilde{\dot{q}}_i = \tilde{\dot{q}}_{in} - \tilde{\dot{q}}_{out} = Y_t\tilde{T}_o - Y_s\tilde{T}_i. \tag{10}$$

To better understand superposition and the usefulness of this approach, it is worthwhile at this point to explore the steady state case. As will be shown below, for the steady state case: $Y_t = Y_s = U$, where U is the steady state U-value (i.e. steady state thermal conductance) for the wall and $U = 1/R$, where R is the corresponding steady state thermal resistance of the wall (including the convection surface resistances) [10]. Utilizing the superposition approach for the steady state case, the net heat flow $\bar{\dot{q}}_i$ is now the sum of $\bar{\dot{q}}_{in} = U\bar{T}_o$ and $\bar{\dot{q}}_{out} = U\bar{T}_i$ and hence:

$$\bar{\dot{q}}_i = U(\bar{T}_o - \bar{T}_i) = (\bar{T}_o - \bar{T}_i)/R \tag{11}$$

(noting that $\bar{\dot{q}}_{out}$ and $\bar{\dot{q}}_{in}$ have opposite signs). Clearly this is the expected result for the steady state case - a result usually obtained utilizing the thermal equivalent of Ohms law [10]. However it further demonstrates the usefulness of the superposition approach in determining heat flows, especially when considering the more complex case of periodic heat flows as described by Eqn. 10.

1.4. Transfer and self-admittance for typical building materials

Shown in Fig. 3 are the magnitudes of the transfer and self-admittance values of some typical building materials as a function of the period in days. Surface resistance values of R_{so} = 0.04 m^2K/W and R_{si} = 0.13 m^2K/W were used in the calculations as well as the thermal properties given in Table 1. The two vertical dashed lines in Fig. 3 correspond to yearly and daily periods. Note that for all materials the yearly values of $|Y_t|$ and $|Y_s|$ are equal to each other and also to the total U-value (i.e., including the surface resistances). That is, the yearly periodic response of the materials is the same as the steady state case. This is also true for the glass and insulation for a daily period. However for the brick wall for a daily period, $|Y_t|$ has decreased while $|Y_s|$ has increased from the steady state value. At very short periods, for all materials $|Y_t|$ tends towards zero whilst $|Y_s|$ tends towards a value of $1/R_{si}$ (7.7 $W/m^2.K$). That is all materials for low periods (high frequency) act as a short circuit and $|Y_s| \rightarrow 1/R_{si}$. That is the internal surface resistance is the only impedance to heat flow (in response to $\tilde{T}_i$ with $\tilde{T}_o$= 0).

Table 1. Properties of building materials used for calculations for Fig. 3 [10].

Property	Brick	Insulation (Expanded polyurethane)	Glass
Thickness, L m	0.2	0.1	0.003
Density, ρ kg/m^3	1920	24	2220
Thermal conductivity, k W/m.K	0.90	0.023	1.38
Specific heat c_p J/kg.K	790	1600	708
R-value (material only) $m^2.K/W$	0.11	4.35	0.0022

The above observations are useful in providing a better general understanding of periodic heat flow through building materials. Thin or lightweight materials such as glass and insulation have heat transfer properties which can, to a fairly good approximation, be described utilising their steady state U value for steady state, yearly and daily periods. For heavier weight materials such as bricks and concrete, the daily thermal response requires a detailed calculation of admittances, however U-values are sufficient to describe the yearly response.

As such it is useful to examine a building's thermal response to a simplified external temperature that is made up of only three Fourier components, that is a constant, a yearly and a daily period. Shown in Table 2 are the admittances for two walls which from the outside surface consists of 100 mm of insulation (expanded polyurethane) and either 100 mm or 200 mm of brick. The "lag" is also given for Y_t and Y_s (i.e., the time difference between the peak of the periodic excitation temperature and the peak of the heat flow [3,8]). The other thermal properties are given in Table 1. Note that the admittances are the daily values and that the yearly values are not given as they are identical to the steady state *U*-value.

For both walls, the addition of insulation to the brick wall significantly improves the *U*-value of the wall in comparison to the brick only wall (see Fig. 3). Note that for the thicker wall, the additional 100 mm of brick only marginally improves the *U*-value of the wall, however the daily transfer admittance is significantly reduced and the daily self-admittance decreases only slightly. Some researchers and designers would advocate wall thicknesses that maximize Y_s (see for example [3]), however the benefit of decreasing Y_t should also be considered.

It is useful to consider the thermal response of both walls for a simple case with $\tilde{T}_i = 0$ (i.e., T_i is constant as would be essentially the case for an air-conditioned room). Consider an outside temperature with a constant value of 18°C, plus a yearly and daily frequency component with angular frequencies of ω_y and ω_d respectively both with an amplitude of 5°C. Hence T_o is given by:

$$T_o = 18 + 5cos(\omega_y t) + 5cos(\omega_d t) \tag{12}$$

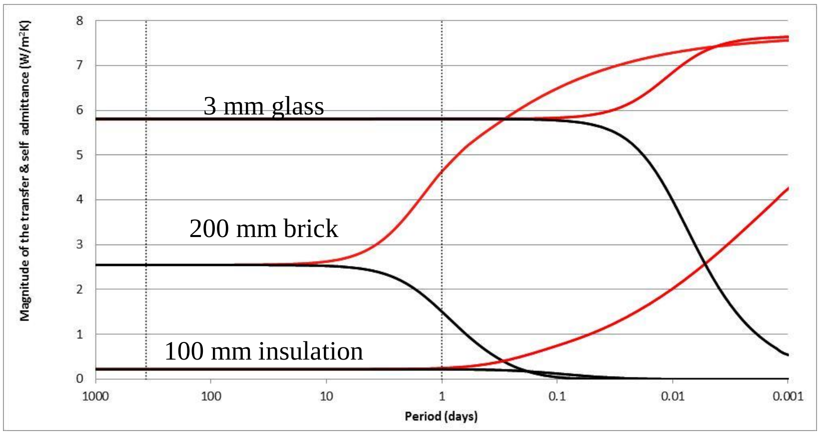


Fig. 3. Magnitude of the transfer admittance (black curves) and self-admittance (red-curves) for different building materials as a function of the period.

Table 2. Heat transfer properties for two composite walls. Admittance values are for *P* = 24 h.

Wall description	*U* value (W/m².K)	$\lvert Y_t \rvert$ (W/m².K)	Y_t Lag (hours)	$\lvert Y_s \rvert$ (W/m².K)	Y_s Lag (hours)
Wall # 1: 100 mm insulation / 100 mm brick	0.22	0.10	-5.6	5.18	+1.9
Wall #2: 100 mm insulation / 200 mm brick	0.21	0.04	-8.5	4.86	+1.2

Considering only the constant and yearly terms then the maximum (i.e. summer) and minimum (i.e. winter) values are 23°C and 13°C. These temperatures are approximately the daily average temperatures for Sydney in December and July respectively. In addition typical daily temperature swings in Sydney have an amplitude of 5°C giving typical daily summer maximums and minimums of 28°C and 18°C respectively. In winter typical daily maximums and minimums are 18°C and 8°C respectively [11]. Shown in Table 3 are the steady state heat flows for a constant outside temperature of 23°C (summer) and 13°C (winter), as well as the amplitude of the daily periodic heat flows for the two composite walls (area 140 m² and thermal properties from Table 2) and two glazing options with an area of 40 m^2. Calculations were carried out with a fixed indoor temperature of $T_i = 22$ºC and utilizing Eqns. 10 and 11.

For these two walls under typical outdoor temperatures for Sydney the steady state heat flows are greatest in winter (due to the greater temperature difference between inside and outside) however all heat flow values are relatively small for 140 m^2 of wall. For the thicker wall (#1) – the daily heat flow amplitude is almost negligible. In comparison for the 40 m^2 of glazing, the single glazed windows have significant heat flows whilst double glazed windows have heat flows closer to those of the walls.

Table 3. Steady state heat flow and amplitude of daily periodic heat flows for two composite walls of area 140 m^2 (properties from Table 2) and two glazing options with an area of 40 m^2.

Element	Season	$\bar{Q}_i$ (W)	$\lvert\tilde{Q}_i\rvert$ (W)	Element	Season	$\bar{Q}_i$ (W)	$\lvert\tilde{Q}_i\rvert$ (W)
Wall #1	Winter	-272	72	Single glazing	Winter	-2160	1200
	Summer	+31	72	(U = 6 W/m^{2}K)	Summer	+240	1200
Wall #2	Winter	-266	30	Double glazing #2	Winter	-540	300
	Summer	+29	30	(U = 1.5 W/m^{2}K)	Summer	+60	300

1.5. Indoor temperature for a room with different wall to window ratios

Utilising the admittance approach it is straight forward to calculate the free running internal temperature of a room. This will be illustrated in this section for a simple case, considering only heat flow through the walls and glazing of a single room and considering the impact of the outdoor temperature only i.e., walls and glazing shaded. In this case the net heat flows into the indoor temperature node are given by:

$$(A_w T_o Y_{tw} - A_w T_i Y_{sw}) + \left(A_g T_o Y_{tg} - A_g T_i Y_{sg}\right) = 0 \qquad (13)$$

where the first term in brackets represents the net heat flow through the walls and the second term in brackets represents the net heat flow through the glazing. Conservation of energy requires that the sum of all heat flows through a node must be zero (the thermal equivalent of "Kirchhoff's current rule" [9]).

Re-arranging for T_i gives:

$$T_i = T_o\left(\frac{A_w Y_{tw} + A_g Y_{tg}}{A_w Y_{sw} + A_g Y_{sg}}\right) \qquad (14)$$

For the steady state case, as all admittances are equal to their corresponding U-values, then the term in brackets is unity and the steady state indoor temperature is simply given by $\bar{T}_i = \bar{T}_o$. This is a useful result: in the absence of solar or internal gains to a room the average indoor temperature will be equal to the average outdoor temperature.

Adding the daily periodic case, Fig. 4 shows the indoor temperate T_i as a function of the window to wall ratio (WWR) for ratios ranging from 0 to 90%. The outdoor temperature is also plotted with an average value of 13ºC and a daily amplitude of 5ºC. The thermal properties of the wall are those reported in Table 2 (wall #2) and the windows have a U-value of 6 W/m^2.K (and for the windows: $U = Y_t = Y_s$, for the yearly and daily Fourier components).

Note that the indoor temperature lag and amplitude decrement are strongly dependent on the window to wall ratio. For a room with a WWR = 0 the indoor temperature is strongly damped and is essentially the average of the outdoor temperature (13ºC). Note that it has a temperature lag of -7.5 hours (i.e. the peak of $\tilde{T}_i$ occurs 7.5 hours after the peak of $\tilde{T}_o$). For a WWR = 0, then $\tilde{T}_i = \tilde{T}_o(Y_{tw}/Y_{sw})$ and the ratio of the admittances has a lag of -7.5 hours. For a WWR = 20% the lag is -1.8 hours. Note from Table 2, that the lag associated with Y_t is -5.6 hours. Often lag is discussed in the literature and the focus is on lag associated with Y_t. However as discussed above temperature lags in a building require an understanding of Y_s as well as Y_t and very importantly, the WWR. As glazing transfers significant heat in and out of a building with essentially zero lag in response to daily outdoor temperature swings, the WWR has a significant impact on indoor temperature amplitude and lag.

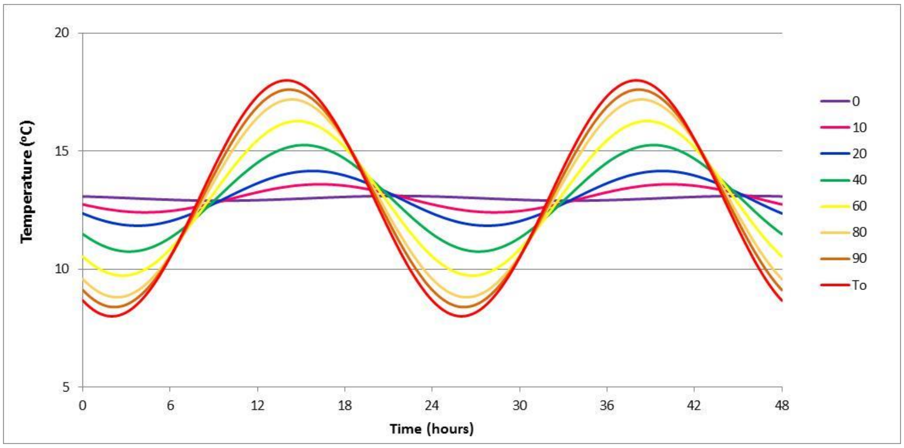


Fig. 4. Indoor temperature for a room with varying window to wall ratio (0% – 90%).

2. Calculation of the periodic internal temperature of a building using admittances

Shown in Fig. 5 is a circuit representation of a building that will be used to demonstrate the admittance approach more fully. The various elements of a building (walls, windows, floors etc.) can be represented using complex thermal impedances, Z, which are the periodic equivalent of a thermal resistance. For simplicity two impedances are shown to graphically illustrate i) the opaque elements, Z_{opaque} and ii) the glazed elements of the building, $Z_{glazing}$. The admittance Y of each element is simply the inverse of the thermal impedance. Infiltration is modeled using a thermal impedance, Z_{infil}. Other periodic heat sources such as solar gains through the glazing, $\dot{Q}_{solar}$ and internal gains, $\dot{Q}_{int}$ are also included. The air inside a building is modeled as a single thermal capacitor, C_{air}. Two periodic external temperature sources are shown. The sol-air temperature source T_{sa} represents the Thevenin equivalent temperature associated with the external ambient temperature and the incident solar radiation on opaque elements of a building [2, 12]. The outdoor air temperature T_o is connected to the thermal impedances associated with the windows and infiltration. Note that Fig. 5 allows the analysis of both the steady state and periodic cases.

Firstly the periodic case will be examined. Using the superposition principle, as described previously, allows very simple expressions to be derived for $\tilde{T}_i$. Heat flows through the internal temperature node will be considered, as before for two cases: 1) heat flowing *out* of the node due to the indoor temperature $\tilde{T}_i$ with all other independent sources set to zero and 2) heat flowing *into* the node due to all the other sources with $\tilde{T}_i$ set to zero.

For the first case, the heat flow $\tilde{\dot{Q}}_{out}$ due to the periodic indoor temperature $\tilde{T}_i$ with all other temperature and heat sources set to zero is:

$$\tilde{\dot{Q}}_{out} = \tilde{T}_i\left(\sum_k A_k Y_{s_k} + C_{infil} + Y_{air}\right) \tag{15}$$

The first term in the equation is a sum of heat flows at the internal surfaces of ***all*** the building elements such as walls, glazing, ceiling and floor with area A_k and self-admittance Y_{sk}. The second term is heat flow due to infiltration (with thermal conductance of $C_{infil} = 1/3 N_{ach} V$) [8] and the final term is heat flow associated with the room air with admittance $Y_{air} = j\omega C_{air} V$, where N_{ach} is the number of air changes per hour, V is the volume of the building or zone, and C_{air} is the volumetric heat capacity of the air [J/m^{3}K].

Similarly for the second case, the heat flow $\tilde{\dot{Q}}_{in}$ due to all the other sources, with $\tilde{T}_i$ zeroed is:

$$\tilde{\dot{Q}}_{in} = \tilde{T}_o\left(\sum_i A_i Y_{t_i} + C_{infil}\right) + \sum_j \tilde{T}_{sa_j} A_j Y_{t_j} + \tilde{\dot{Q}}_{solar} + \tilde{\dot{Q}}_{int} \tag{16}$$

The first term is the sum of heat flows at the internal surfaces all of the glazing elements with area A_i and transfer admittance Y_{t_i}, in response to the external air temperature $\tilde{T}_o$ plus the infiltration heat flow. The second term is the sum of heat flows at the internal surfaces of all the opaque building elements such as walls, ceiling and floor with area A_j and transfer-admittance Y_{t_j}, in response to the external sol-air temperature $\tilde{T}_{sa_j}$ incident on the j^{th} element. The last two terms describe the periodic heat gains into the building due to solar gain through glazing, $\tilde{\dot{Q}}_{\text{solar}}$ and internal heat gains, $\tilde{\dot{Q}}_{\text{int}}$.

Superposition can now be used to determine the net rate of heat flow into and out of the internal temperature node for the complete thermal circuit (see Fig. 5).

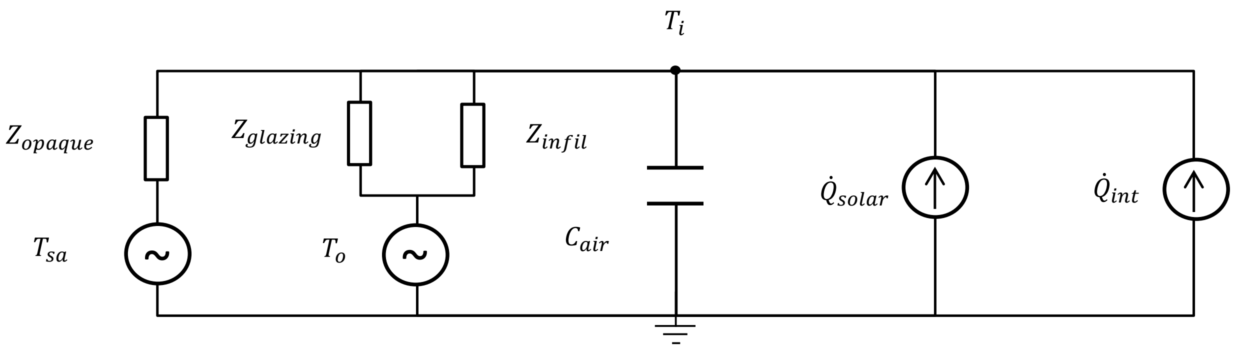


Fig. 5. A thermal circuit representation of a building.

That is the total heat flow in and out of the internal temperature node is the sum of heat flows from cases 1 and 2. For a single node, using "Kirchhoff's current rule", the sum of all heat flowing into and out of the node for all pathways must sum to zero [9]. This is a statement of the conservation of energy. Noting that the heat flows $\tilde{\dot{Q}}_{in}$ and $\tilde{\dot{Q}}_{out}$ have opposite signs, then

$$\tilde{\dot{Q}}_{in} - \tilde{\dot{Q}}_{out} = 0 \tag{17}$$

Hence substituting Eqns. 15 and 16 into Eqn. 17 and solving for $\tilde{T}_i$ yields:

$$\tilde{T}_i = \frac{\tilde{T}_o\left(\sum_i A_i Y_{t_i} + C_{infil}\right) + \sum_j \tilde{T}_{sa_j} A_j Y_{t_j} + \tilde{\dot{Q}}_{solar} + \tilde{\dot{Q}}_{int}}{\left(\sum_k A_k Y_{s_k} + C_{infil} + Y_{air}\right)} \tag{18}$$

This equation allows the calculation of the free running, periodic, internal temperature of a building. Inputs required are external climate variables (sol-air and air temperatures, and solar gains) and internal gains into the building as well as the self and transfer admittances for all elements of the building. Inputs can be represented using Fourier series with sufficient frequency terms to ensure accurate representation of the heat sources (e.g. temperatures and internal heat gains). Note that for each frequency in the Fourier series, a corresponding admittance value needs to be calculated. Superposition can then be used to sum all contributions to $\tilde{T}_i$ to calculate the total thermal response.

3. Average value of the indoor temperature

A similar analysis can be used to yield the average indoor temperature $\overline{T}_i$. Essentially the circuit is analysed for the "constant" plus the yearly frequency component of the Fourier series representation. Hence in this case, admittance values – both self and transfer admittance, will tend towards the *U* value for the building element. In addition the admittance of the air will be essentially zero. Hence:

$$\bar{T}_i = \frac{\bar{T}_o(\sum_i A_i U_i + C_{infil}) + \sum_j \bar{T}_{sa_j} A_j U_j + \bar{\dot{Q}}_{solar} + \bar{\dot{Q}}_{int}}{(\sum_k A_k U_k + C_{infil})} \quad (19)$$

If we ignore for simplicity the additional sol-air temperature (e.g. shaded opaque elements) then the sol-air temperatures are essentially equal to the outside air temperatures and so we can solve for $\bar{T}_i$ to yield:

$$\bar{T}_i = \bar{T}_o + \frac{\bar{\dot{Q}}_{solar} + \bar{\dot{Q}}_{int}}{(\sum_k A_k U_k + C_{infil})} \quad (20)$$

(Note that $\sum_k A_k U_k = \sum_i A_i U_i + \sum_j A_j U_j$). Hence Eqn. 20 describes a free running building and for the case of shaded external opaque elements, the average internal temperature will be equal to the average ambient temperature plus an additional temperature rise due to solar and internal heat gains.

In summer a building described by Eqn. 20 will potentially be comfortable provided the average ambient temperature is within the comfort zone and solar and internal heat gains are minimised. For a well-insulated building (low *U*- value) – the internal temperature will increase if solar or internal gains are too high. In this case greater ventilation will keep temperatures lower – but again this will only be comfortable if the average outdoor temperature is within a comfortable range. If too warm – then night ventilation would be best.

In winter for climates outside the tropics, the average outdoor temperature will be lower than a comfortable temperature range. Hence best to maximise the solar heat gain, have low *U* value construction, minimise the surface area and minimise infiltration to increase $\bar{T}_i$ above $\bar{T}_o$.

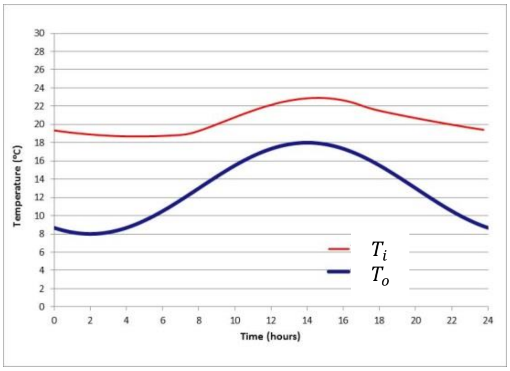
4. Indoor temperature – free running building with solar gain

Shown in Fig. 6 is the calculated indoor temperature for a building in a) winter with solar gain through a Northern window and b) summer assuming zero solar gains (i.e., all windows are externally shaded) . A limited number of inputs are required for such a model and will be given below. The building is located in Sydney (latitude 34^{o}S), and the dates considered are July 21st and Dec 21st. The building is rectangular with a floor area of 20 m x 10 m, with the long side facing North and the ceiling height is 3 m. The walls have thermal properties given in Table 2 and wall #2 is utilised. For simplicity the roof and floor are considered massless and have an *R* value of 5 m^{2}K/W. The infiltration rate is N_{ach} = 0.1 ACH and the average internal gains = 1 W/m^2. The glazing has a *U*-value = 1.5 W/m^2.K and SHGC = 0.7. The WWR is 26% for the North facing wall, and 20% for all other walls. Solar insolation is assumed to occur only on the Northern windows with a value of 4.6 kWh/m^2 per day (clear sky model [13]). For Fig. 6(a), the average outside temperature is $\bar{T}_o$ = 13^{o}C corresponding to July in Sydney with a 5 degree daily amplitude. For Fig. 6(b), $\bar{T}_o$ = 23^{o}C corresponding to Decmeber in Sydney with a 5 degree daily amplitude.

For the parameters given above the building has $\bar{T}_i =$ 20.5^{o}C with a temperature swing of 2.1^{o}C in winter. For a residential building in Sydney in winter this would be a comfortable indoor temperature range. Similarly for summer, then $\bar{T}_i =$ 24.1^{o}C with a swing of only 0.9^{o}C. Again this would be a comfortable temperature range in Sydney in the summer. The important point here is that the average indoor temperature calculation using Eqn. 20 is a relatively simple, steady state calculation; however it provides significant insight in to the thermal performance of a building. To illustrate this, the thermal gains and building fabric gains and losses over 24 hours are shown in Table 4.

Table 4. Thermal gains and building fabric gains/losses (kWh) over 24 hours.

Season	Solar gains	Internal gains	Walls	Glazing	Ceiling	Floor	Infiltration
Winter	29.2	4.8	-5.3	-10.6	-7.2	-7.2	-3.6
Summer	0	4.8	0.8	1.5	1.0	1.0	0.5



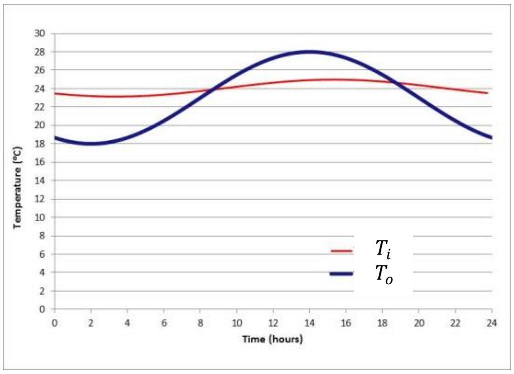


Fig. 6. Indoor and outdoor temperatures for a free running building for a typical day in Sydney a) winter and b) summer.

5. Conclusions

This paper has presented an overview and exploration of the admittance approach for describing periodic heat flow through building elements. The admittance model used here is quite simple to program and was implemented in Excel. The simplified model allows many "what if" questions to be asked quickly and easily as there are very few input parameters, yet they capture the essential fundamentals required to describe a building. However the important point to note from the results presented here is that valuable insight into the performance of a building can be determined from steady state calculations. If further insight of the periodic components of temperature or heat flow is required, the admittance method allows good insight to be gained. Of course there are a number of simplifications made in this approach. For example internal radiation exchanges inside the building are greatly simplified. However the approach presented here allows researchers and practitioners to gain a far better feeling and understanding of periodic heat flows in a building.

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